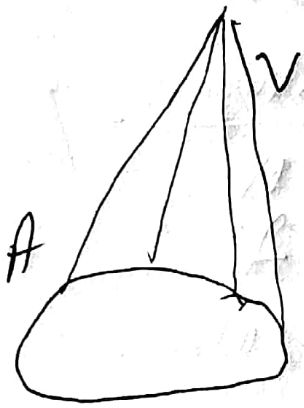
B.Sc (Maths) — part I
paper — II

The Cone (3D Geometry)

Cone :- Def : — A cone is a surface generated by a straight line which passes through a fixed point and intersects a given curve.

The fixed point V is called vertex, the moving line VA is called generator and the fixed curve is called the guiding curve.



Right circular cone :- A right circular cone is a surface generated by a line which passes through a fixed point called the vertex and makes a const. angle with the fixed line through the vertex. In right circular cone the guiding curve is a circle and hence name right circular cone.

Theorem :- To find the equation of conic whose vertex is (α, β, γ)

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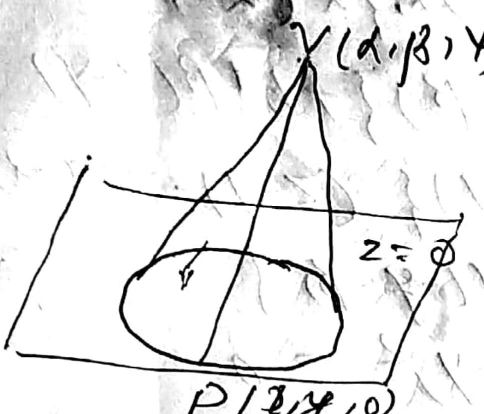
and base the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0, \quad z=0$$

Soln —



Let the equation of line through the vertex $V(\alpha,\beta,\gamma)$ of the cone is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{—(1)}$$

If it meets the plane $z=0$ at the point P then from (1) $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{-\gamma}{n}$

$$x=\alpha-\frac{l}{n}\gamma, \quad y=\beta-\frac{m}{n}\gamma$$

The point $P(\alpha-\frac{l}{n}\gamma, \beta-\frac{m}{n}\gamma, 0)$ will lie on the guiding curve $ax^2+2hxy+by^2+2gx+2fy+c=0, z=0$

if $a(\alpha-\frac{l}{n}\gamma)^2+2h(\alpha-\frac{l}{n}\gamma)(\beta-\frac{m}{n}\gamma)+b(\beta-\frac{m}{n}\gamma)^2+2g(\alpha-\frac{l}{n}\gamma)+2f(\beta-\frac{m}{n}\gamma)+c=0$ —(2)

Now eliminating l, m, n from (1) and (2) we have

$$a\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)^2+2h\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)+b\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)^2$$

$$+2g\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)$$

$$+2f\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)+c=0$$

or $a\{\alpha(z-\gamma)-\gamma(x-\alpha)\}^2+2h\{\alpha(z-\gamma)-\gamma(x-\alpha)\}\{\beta(z-\gamma)-(y-\beta)\}+b\{\beta(z-\gamma)-\gamma(y-\beta)\}^2+2g(z-\gamma)\{\alpha(z-\gamma)-\gamma(x-\alpha)\}+2f(z-\gamma)\{\beta(z-\gamma)-\gamma(y-\beta)\}$

$$+c(z-\gamma)^2=0$$

$\therefore\ a(\alpha z-\gamma x)^2+2h(\alpha z-\gamma x)(\beta z-\gamma y)+b(\beta z-\gamma y)^2+2g(z-\gamma)(\alpha z-\gamma x)+2f(z-\gamma)(\beta z-\gamma y)+c(z-\gamma)^2=0$

This is required eqn. of the Cone.

<u>Theorem</u> To find the Condition that the general equation of second degree in x, y, z may represent a Cone.

Soln: — ~~The~~

The general equation of the second degree in x, y, z is

$$F(x,y,z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0 \quad \text{— (1)}$$

If (1) represents a Cone. Let (α,β,γ) be the Co-ordinates of it vertex shifting the origin (α,β,γ) the eqn (1) is transformed

$a(x+\alpha)^2+b(y+\beta)^2+c(z+\gamma)^2+2f(y+\beta)(z+\gamma)+2g(z+\gamma)(x+\alpha)+2h(x+\alpha)(y+\beta)$
$+2u(x+\alpha)+2v(y+\beta)+2w(z+\gamma)+d=0$

& $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$
$+2\{x(a\alpha+h\beta+g\gamma+u)+y(h\alpha+b\beta+f\gamma+v)+z(g\alpha+f\beta+c\gamma+w)\}+2F(\alpha,\beta,\gamma)=0$ —— (2)

Since eqn (2) represents a cone with vertex at the origin therefore (2) must be reduced to a homogeneous eqn.. For this

$a\alpha+h\beta+g\gamma+u=0$ —— (3)
$h\alpha+b\beta+f\gamma+v=0$ —— (4)
$g\alpha+f\beta+c\gamma+w=0$ —— (5)
and $F(\alpha,\beta,\gamma)=0$

eqn. (6) can be written as
$\alpha(a\alpha+h\beta+g\gamma+u)+\beta(h\alpha+b\beta+f\gamma+v)+\gamma(g\alpha+f\beta+c\gamma+w)+(u\alpha+v\beta+w\gamma+d)=0$

From (3) (4) and (5) we get
$u\alpha+v\beta+w\gamma+d=0$ $\longrightarrow$ (7)

Now eliminating α,β,γ between (3) (4) (5) and (7) we get

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}$$

which is the required condition